Statistical Inference for Unreliable Queueing Systems

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Introduction

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- identifiability results for parameters
- a unified framework for the construction of estimators and their accuracy
- development of universal statistical tests

Outline of the Talk

- nonparametric estimation of model parameters for $M/G/\infty$ systems

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- presentation of a universal method based on a sequence of differences

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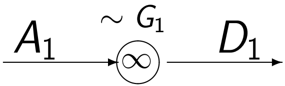
- presentation of a universal method based on a sequence of differences
- statistical analysis of unreliable systems where the service processes are randomly interrupted

Nonparametric Estimation for M/G/∞

Poisson arrival process $\mathbf{A_1}$ with parameter λ
observations: $\mathbf{A_1}$ and $\mathbf{D_1} = (Y_n^1)_{n\in\mathbb{Z}}$
aim: nonparametric estimation of the service time cdf G_1
(no matching of arrival points to departure points possible)
assumption: system in steady state and $\mathbb{E}G_1 < \infty$



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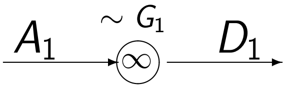

Applications: cell mobility, traffic systems, social migration systems, supply chain networks, communication systems, waiting lines in daily life

Sequence of Differences in Continuous Time (Brown (1970))

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Z_n^1 is defined as the distance from Y_n^1 to the nearest point of A_1 on the left of Y_n^1, for all $n \in \mathbb{Z}$.

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1. *The sequence $(Z_n^1)_{n\in\mathbb{Z}}$ is stationary.*
2. *The sequence $(Z_n^1)_{n\in\mathbb{N}}$ is ergodic.*
3. *We have for the distribution function $H_1(\cdot)$ of Z_1^1 :*

$$H_1(x) = \begin{cases} 1 - e^{-\lambda x}\left(1 - G_1(x)\right) & \text{for } x \geq 0. \end{cases}$$

Estimation of the cdf G_1

We have: $G_1(x) = 1 - e^{\lambda x}(1 - H_1(x))$ for $x > 0$.

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Theorem (Brown (1970))

The sequence of random functions $(\widehat{G}_{1,n})_{n\in\mathbb{N}}$ with

$$\widehat{G}_{1,n}(x) := \sup_{y\in[0,x]} \left\{ 1 - \left(1 - \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(-\infty,y]}(Z_i^1) \right) e^{\lambda y} \right\} \text{ for } x \geq 0$$

converges a.s. uniformly to the true service time cdf G_1 as $n \to \infty$, that is,

$$\mathcal{P}\left(\sup_{x\in\mathbb{R}} |\widehat{G}_{1,n}(x) - G_1(x)| \xrightarrow{n\to\infty} 0 \right) = 1.$$

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The proof uses Birkhoff's ergodic theorem.

Extension of Brown's Approach

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Then the crucial relation is:

$$H^{(r)}(x) = 1 - (1 - G_1(x))e^{-\lambda x}\frac{(\lambda x)^{r-1}}{(r-1)!} - \sum_{j=0}^{r-2}\frac{e^{-\lambda x}(\lambda x)^j}{j!}$$

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Idea for applications: Choose r adaptively.
In case of $\lambda > \mathbb{E}G$, a strong improvement of Brown's method can be observed.

Further Statistical Work for M/G/∞ Nodes

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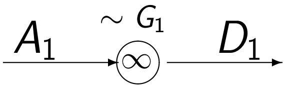
- Langrock & W. (2012):
  extends the approach of Brillinger (1974) to feedforward networks

Sequence of Differences in Discrete Time

Edelmann & W. (2014)

arrival process $\mathbf{A_1} = (A_1(t))_{t\in\mathbb{Z}}$ is i.i.d. sequence (batch arrivals)
observations: $\mathbf{A_1}$ and $\mathbf{D_1}$ starting $t = 0$
aim: nonparametric estimation of the service time cdf G_1
assumptions:
$\mathbb{E}(A_1(0)) < \infty, \mathbb{E}G_1 < \infty, G_1(0) = 0$ and $c_1 := \mathbb{P}(A_1(0) = 0) > 0$.



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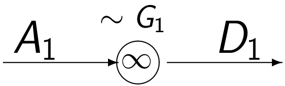

We define the *sequence of differences* $(Z_1(t))_{t\in\mathbb{Z}_1}$ as

$$Z_1(t) := t - \max\{n < t | A_1(n) > 0\} \text{ for } t \in \mathbb{Z}.$$

CRUCIAL RELATION

The following relation can be shown:

$$\frac{\mathbb{E}\left[D_1(0)\mathbf{1}_{\{Z_1(0)\leq x\}}\right]}{\mathbb{E}[D_1(0)]} = 1 - c_1^x\left(1 - G_1(x)\right) \textit{ for all } x \in \mathbb{N}.$$

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It can be written as

$$\begin{aligned}\mathbb{E}\left[D_1(0)\mathbf{1}_{\{Z_1(0)>x\}}\right] &= \mathbb{E}[D_1(0)](1 - G_1(x)) \cdot c_1^x \\ &= \mathbb{E}\left[D_1(0)|Z_1(0) > x\right] \cdot \mathbb{P}(Z_1(0) > x).\end{aligned}$$

$\Longrightarrow$ Interpretation as a conditional expectation

Definition of the Estimator

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Set $H(x) := \frac{\mathbb{E}\left[D_1(0)\mathbf{1}_{\{Z_1(0)\leq x\}}\right]}{\mathbb{E}[D_1(0)]}$ and $\widehat{H}_n(x) := \frac{\sum_{i=1}^n D_1(i)\mathbf{1}_{\{Z_1(i)\leq x\}}}{\sum_{i=1}^n D_1(i)}$

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Theorem (Edelmann & W. (2014))

Under the assumptions of the model $(\widehat{G}_n(\cdot))_{n\in\mathbb{N}}$ converges a.s. uniformly to G_1 as $n \to \infty$.

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In Edelmann & W. (2014) the result is further extended to stochastic networks of general topology.

Notations

$$\widehat{G}_n(x) := 1 - \hat{c}_n^{-x}\left(1 - \widehat{H}_n(x)\right) \text{ for all } x \in \mathbb{N}.$$

Denote

$$\mathcal{G} := \left(G_1(k)\right)_{k\in\mathbb{N}}, \mathcal{G}_n := \left(\widehat{G}_n(k)\right)_{k\in\mathbb{N}}, \mathcal{H} := \left(H(k)\right)_{k\in\mathbb{N}}, \mathcal{H}_n := \left(\widehat{H}_n(k)\right)_{k\in\mathbb{N}}.$$

Aim: Proof of a functional clt for $\sqrt{n}\,(\mathcal{G}_n - \mathcal{G})$.

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Aim: Proof of a functional clt for $\sqrt{n}\,(\mathcal{G}_n - \mathcal{G})$.

We consider the separable Banach space c_0 of all sequences $x = (x_k)_{k\in\mathbb{N}}$, which converge to zero for $k \to \infty$.
Norm on c_0 : $\|x\| = \sup_{k\in\mathbb{N}} |x_k|$.
(cf. Henze (1996)).

Functional Central Limit Theorem

Theorem (Schweer & W. (2015))

Let in addition $\mathbb{E}[A(0)^2] < \infty$ *and* $\mathbb{E}G^{2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$. *Then there is a Gaussian process* $\mathcal{V} = (V_k)_{k\in\mathbb{N}}$ *in* c_0, *such that*

$$\sqrt{n}\,(\mathcal{G}_n - \mathcal{G}) \xrightarrow{\mathcal{D}} \mathcal{V}$$

It follows: $\mathbb{E}\left[V_k\right] = 0$ *and*

$$\begin{aligned}\mathbb{E}\left[V_k V_m\right] &= \frac{\tau_{k,m}^2}{c^{k+m}} + km(1-H(k))(1-H(m))\frac{1-c}{c^{k+m+1}} \\ &\quad - k(1-H(k))\frac{\tau_{1,m}^2}{c^{k+m+1}} - m(1-H(m))\frac{\tau_{1,k}^2}{c^{k+m+1}},\end{aligned}$$

Method of Proof

$$\widehat{G}_n(x) := 1 - \hat{c}_n^{-x}\left(1 - \widehat{H}_n(x)\right) \text{ for all } x \in \mathbb{N}.$$

- show weak convergence of $\sqrt{n}\,(\mathcal{H}_n - \mathcal{H})$ in c_0.
  For this we have to show:
  (1) weak convergence of the fidis using a clt for stationary and ergodic sequences (Billingsley (1999))
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- apply the functional delta method with Hadamard differentiability to show weak convergence of $\sqrt{n}\,(\mathcal{G}_n - \mathcal{G})$ in c_0.

Unreliable M/G/∞ Nodes

Poisson arrival process A_1 with intensity λ
service time X with cdf $G_1 = \delta_{x_0}$ (deterministic)
up time of each server $\sim \exp(\lambda_c)$
repair time R of each server $\sim$ cdf F_R

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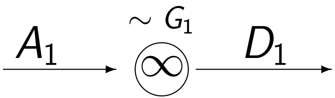
Observations: the jumps of A_1 and D_1 in $(0, T]$
Aim: nonparametric estimation of repair time cdf F_R and of λ_c



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repair time R of each server $\sim$ cdf F_R

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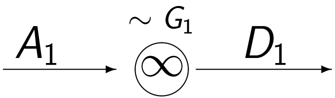
Observations: the jumps of A_1 and D_1 in $(0, T]$
Aim: nonparametric estimation of repair time cdf F_R and of λ_c



Using Brown's method a uniformly strongly consistent estimator for the cdf G_{soj} of the sojourn time in the system can be constructed.

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in terms of random variables:

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Lemma (W. & Kocamer (2021))

The sequence $(\lambda_{c,n})_{n \in \mathbb{N}}$ converges a.s. to the true break down rate λ_c.

Decompounding Approach
(Buchmann & Grübel (2003))

We have

$$G_{\mathrm{soj}}(z) = \sum_{k=0}^{\infty} e^{-\lambda_c x_0} \frac{(\lambda_c x_0)^k}{k!} F_R^{*k}(z - x_0), \quad z \in \mathbb{R}_+.$$

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where $G_{\mathrm{soj}}^0(\cdot + x_0) := G_{\mathrm{soj}}(\cdot + x_0) - G_{\mathrm{soj}}(x_0)$.

Formal Framework

Denote by

- $D[0,\infty)$ the space of càdlàg functions f with $\lim_{x\to\infty} f(x)$ exists, equipped with norm $||f||_\infty := \sup_{x\geq 0} |f(x)|$

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Theorem (W. & Kocamer (2021))

Let $\lambda_0, \tau > 0$ *and* G_0 *be a cdf with* $\tilde{G}_0^0(\tau) < e^{-\lambda_0 x_0}$. *Then there exists an open interval* $J \subset (0, \infty)$ *with* $\lambda_0 \in J$ *and* $\varepsilon > 0$ *such that*

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(A) $\Lambda(\lambda_c, G_{soj}) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1} e^{\lambda_c x_0 k}}{\lambda_c x_0 k} (G_{soj}^0(\cdot + x_0))^{*k}(\cdot)$ *converges in* $D(\tau)$ *for all* $\lambda_c \in J$ *and cdfs* G_{soj} *with* $||G_{soj} - G_0||_\infty < \varepsilon$.

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Idea of the Proof for (A)

The crucial point is:

$$||g * H||_{\infty,\tau} \leq ||g||_{\infty,\tau} \tilde{H}(\tau)$$

for all $g \in D(\tau)$, $\tau > 0$ and all monotone increasing $H \in \bigcup_{\tau>0} D(\tau)$ with finite variation on finite intervals.

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$$G_{\mathrm{soj}}(\cdot) = \sum_{k=0}^{\infty} e^{-\lambda_c x_0} \frac{(\lambda_c x_0)^k}{k!} F_R^{*k}(\cdot - x_0)$$

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Now we use $\log(1 + r) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{r^k}{k}$ for $r \in (-1, 1]$. This renders the same structure as in (1) and the condition $\tilde{G}^0_0(\tau) < e^{-\lambda_0 x_0}$ in the theorem.

Application of the Theorem

Let the conditions of the theorem be fulfilled.
Let $G_{\mathrm{soj,n}}$ be an estimator for G_{soj} which is uniformly strongly consistent. And consider our estimator $\lambda_{c,n}$ for λ_c. Then:

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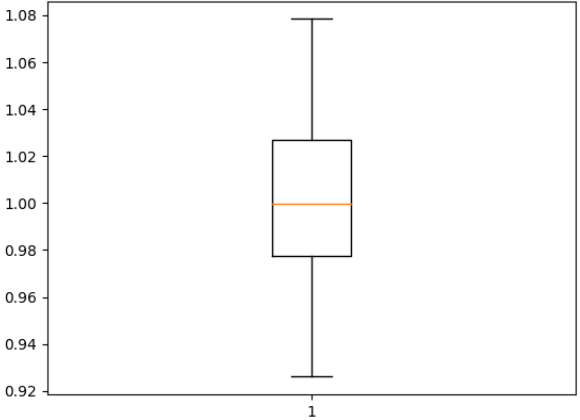
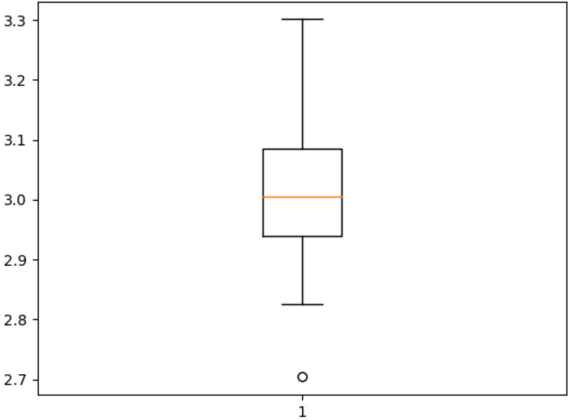
Thus, our estimator for F_R is uniformly strongly consistent in the space $D(\tau)$.

SIMULATIONS - ESTIMATION OF λ_c (1)



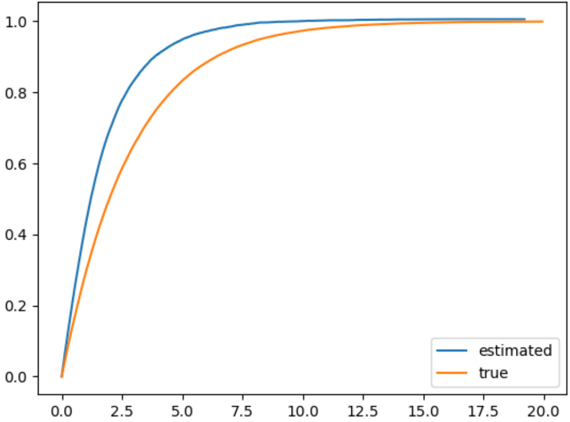
based on 100 simulations of a system with arrival rate $\lambda = 1$, breakdown rate $\lambda_c = 1$, repair time distribution exp(10) and $x_0 = 0.5$

Simulations - Estimation of λ_c (2)



based on 100 simulations of a system with arrival rate $\lambda = 2$, breakdown rate $\lambda_c = 3$, repair time distribution exp(8) and $x_0 = 0.25$

Simulations - Estimation of F_R (1)



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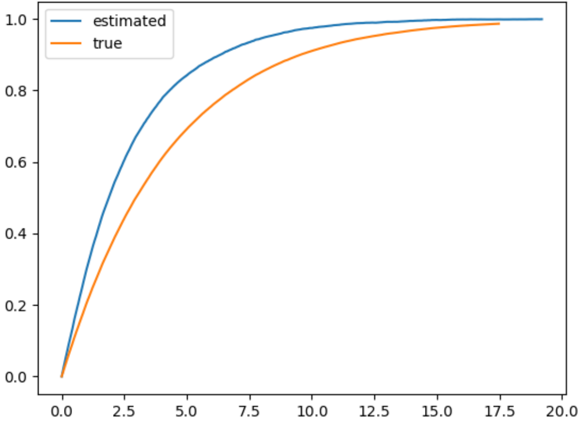

based on a simulation of a system with arrival rate $\lambda = 1$, breakdown rate $\lambda_c = 1$, repair time distribution exp$(1/3)$ and $x_0 = 0.8$

Unreliable M/G/∞ Nodes - General Case

Poisson arrival process A_1 with intensity λ
service time X with general cdf G_1 with $\mathbb{E}G_1 < \infty$ (known)
up time of each server $\sim \exp(\lambda_c)$
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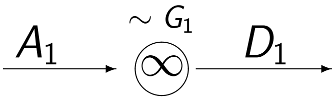

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$$G_{\text{soj}}(z) = \int_0^\infty \sum_{k=0}^\infty e^{-\lambda_c s} \frac{(\lambda_c s)^k}{k!} F_R^{*k}(z-s)\, dG_1(s),\ z \in \mathbb{R}_+.$$

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If a minimum $s_0 > 0$ of the support of G_1 exists and is known, λ_c is identifiable due to

$$G_{\mathrm{soj}}(s_0) = e^{-\lambda_c s_0}.$$

Let us assume here that λ_c is known.

General Decompounding Approach

We have

$$G_{\mathrm{soj}}(z) = \int_0^\infty \sum_{k=0}^\infty e^{-\lambda_c s} \frac{(\lambda_c s)^k}{k!} F_R^{*k}(z-s)\, dG_1(s), \; z \in \mathbb{R}_+.$$

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We compute the Laplace transform:

$$\tilde{G}_{\text{soj}}(\tau) = \sum_{k=0}^\infty w_k(\tau)(\tilde{F}_R(\tau))^k,$$

where

$$w_k(\tau) = \frac{(\lambda_c)^k}{k!} \int_0^\infty e^{-\tau s} e^{-\lambda_c s} s^k dG_1(s) = \frac{(\lambda_c)^k}{k!} \mathbb{E}[e^{-(\tau+\lambda_c)X} X^k]$$

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with $X \sim G_1$.
The coefficients $w_k(\tau)$ are all nonnegative and sum up to $\tilde{G}_1(\tau)$.

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Proposition

The coefficients $\pi_l(\tau)$ may be computed from the following recursion scheme:

$$\pi_1(\tau) w_1^{(1)}(\tau) = 1$$

$$\pi_1(\tau) w_k^{(1)}(\tau) + \pi_2(\tau) w_k^{(2)}(\tau) + \cdots + \pi_k(\tau) w_k^{(k)}(\tau) = 0 \text{ for } k = 2, 3, \ldots,$$

where the $w_k^{(n)}$ are the coefficients of

$$\left(\tilde{G}_{soj}(\tau) - w_0(\tau)\right)^n = \sum_{k=2}^{\infty} w_k^{(n)}(\tau)(\tilde{F}_R(\tau))^k, \; n \geq 2.$$

The proof is based on Henrici (1974), *Applied and computational complex analysis.* Compare also with Bøgsted & Pitts (2010).

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Application of the Theorem

Consider a queueing node with unreliable servers. Assume the exponential breakdown rate λ_c to be known. Let the conditions of the theorem be fulfilled.
Let $G_{\mathrm{soj,n}}$ be an estimator for G_{soj} which is uniformly strongly consistent. Then:

$$\begin{aligned} ||F_{R,n} - F_R||_{\infty,\tau} &= ||\Gamma(G_{\mathrm{soj,n}}) - \Gamma(G_{\mathrm{soj}})||_{\infty,\tau} \\ &\leq \text{ Const } ||G_{\mathrm{soj,n}} - G_{\mathrm{soj}}||_{\infty} \to 0 \text{ for } n \to \infty \; a.s. \end{aligned}$$

Thus, our estimator for F_R is uniformly strongly consistent in the space $D(\tau)$.

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Thank you very much for your attention!

Questions or comments?